

# NOTES ON FORMAL SMOOTHNESS

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*Dedicated to Robert Wisbauer on the occasion of his 65th birthday*

**ABSTRACT.** The definition of an S-category is proposed by weakening the axioms of a Q-category introduced by Kontsevich and Rosenberg. Examples of Q- and S-categories and (co)smooth objects in such categories are given.

## 1. INTRODUCTION

In [12] Kontsevich and Rosenberg introduced the notion of a Q-category as a framework for developing non-commutative algebraic geometry. Relative to such a Q-category they introduced and studied the notion of a *formally smooth object*. Depending on the choice of Q-category this notion captures e.g. that of a *smooth algebra* of [16], which arose a considerable interest since its role in non-commutative geometry was revealed in [8].

The aim of these notes is to give a number of examples of Q-categories, and their weaker version which we term S-categories, of interest in module, coring and comodule theories, and to give examples of smooth objects in these Q-categories. Crucial to the definition of an S-category is the notion of a *separable functor* introduced in [14]. In these notes we consider only the separability of functors with adjoints. This case is fully described by the Rafael Theorem [15]: A functor which has a right (resp. left) adjoint is separable if and only if the unit (resp. counit) of adjunction is a natural section (resp. retraction). For a detailed discussion of separable functors we refer to [7].

Throughout these notes, by a category we mean a set-category (i.e. in which morphisms form sets), by functors we mean covariant functors. All rings are unital and associative. For an  $A$ -coring  $\mathcal{C}$ ,  $\Delta_{\mathcal{C}}$  denotes the coproduct and  $\varepsilon_{\mathcal{C}}$  denotes the counit. Whenever needed, we use the standard Sweedler notation for a coproduct  $\Delta_{\mathcal{C}}(c) = \sum c_{(1)} \otimes_{A\mathcal{C}(2)}$  and for a coaction  $\varrho^M(m) = \sum m_{(0)} \otimes_A m_{(1)}$ .

## 2. SMOOTHNESS AND COSMOOTHNESS IN Q- AND S-CATEGORIES

Here we gather definitions of categories and objects we study in these notes.

**Definition 2.1.** An *S-category* is a pair of functors  $\mathbb{X} = (\bar{\mathfrak{X}} \xrightleftharpoons[u^*]{u_*} \mathfrak{X})$  such that  $u^*$  is separable and left adjoint of  $u_*$ .

This means that in an S-category  $\mathbb{X} = (\bar{\mathfrak{X}} \xrightleftharpoons[u^*]{u_*} \mathfrak{X})$  the unit of adjunction  $\eta : \bar{\mathfrak{X}} \rightarrow u_* u^*$  has a natural retraction  $\nu : u_* u^* \rightarrow \bar{\mathfrak{X}}$ . Therefore, for all objects  $x$  of  $\mathfrak{X}$  and  $y$  of  $\bar{\mathfrak{X}}$ , there exist morphisms

$$\bar{\mathfrak{X}}(y, u^*(x)) \rightarrow \mathfrak{X}(u_*(y), x), \quad g \mapsto \nu_x \circ u_*(g).$$

The notion of an S-category is a straightforward generalisation of that of a Q-category, introduced in [12]. The latter is defined as a pair of functors  $\mathbb{X} = ( \bar{\mathfrak{X}} \begin{smallmatrix} \xrightarrow{u_*} \\ \xleftarrow{u^*} \end{smallmatrix} \mathfrak{X} )$  such that  $u^*$  is full and faithful and left adjoint of  $u_*$ . In a Q-category the unit of adjunction  $\eta$  is a natural isomorphism, hence, in particular, a section. Thus any Q-category is also an S-category. Following the Kontsevich-Rosenberg terminology (prompted by algebraic geometry) the functors  $u_*$  and  $u^*$  constituting an S-category are termed the *direct image* and *inverse image* functors, respectively.

**Definition 2.2.** We say that an S-category  $\mathbb{X} = ( \bar{\mathfrak{X}} \begin{smallmatrix} \xrightarrow{u_*} \\ \xleftarrow{u^*} \end{smallmatrix} \mathfrak{X} )$  is *supplemented* if there exists a functor  $u_! : \bar{\mathfrak{X}} \rightarrow \mathfrak{X}$  and a natural transformation  $\bar{\eta} : \bar{\mathfrak{X}} \rightarrow u^*u_!$ .

In particular, an S-category  $\mathbb{X} = ( \bar{\mathfrak{X}} \begin{smallmatrix} \xrightarrow{u_*} \\ \xleftarrow{u^*} \end{smallmatrix} \mathfrak{X} )$  is supplemented if  $u^*$  has a left adjoint. Furthermore,  $\mathbb{X}$  is supplemented if the functor  $u_*$  is separable, since, in this case, the counit of adjunction has a section which we can take for  $\bar{\eta}$  (and  $u_! = u_*$ ). This supplemented S-category is termed a *self-dual supplemented S-category*.

In a supplemented S-category, for any  $y \in \bar{\mathfrak{X}}$ , there is a canonical morphism in  $\mathfrak{X}$ , natural in  $y$ ,

$$r_y : u_*(y) \rightarrow u_!(y),$$

defined as a composition

$$r_y : u_*(y) \xrightarrow{u_*(\bar{\eta}_y)} u_*u^*u_!(y) \xrightarrow{\nu_{u_!(y)}} u_!(y) .$$

The existence of canonical morphisms  $r_y$  allows us to make the following

**Definition 2.3.** Given a supplemented S-category  $\mathbb{X} = ( \bar{\mathfrak{X}} \begin{smallmatrix} \xrightarrow{u_*} \\ \xleftarrow{u^*} \end{smallmatrix} \mathfrak{X} )$ , with the natural map  $r : u_* \rightarrow u_!$ , an object  $x$  of  $\mathfrak{X}$  is said to be:

- (a) *formally  $\mathbb{X}$ -smooth* if, for any  $y \in \bar{\mathfrak{X}}$ , the mapping  $\mathfrak{X}(x, r_y)$  is surjective;
- (b) *formally  $\mathbb{X}$ -cosmooth* if, for any  $y \in \bar{\mathfrak{X}}$ , the mapping  $\mathfrak{X}(r_y, x)$  is surjective.

*Remark 2.4.* We would like to stress that the notion of formal  $\mathbb{X}$ -(co)smoothness is relative to the choice of the retraction of the unit of adjunction, and the choice of  $u_!$  and  $\bar{\eta}$ , since the definition of  $r$  depends on all these data.

Dually to S- and Q-categories one defines  $S^\circ$ -categories and  $Q^\circ$ -categories.

**Definition 2.5.** An  *$S^\circ$ -category* (respectively  *$Q^\circ$ -category*) is a pair of functors  $\mathbb{X} = ( \bar{\mathfrak{X}} \begin{smallmatrix} \xrightarrow{u_*} \\ \xleftarrow{u^*} \end{smallmatrix} \mathfrak{X} )$  such that  $u^*$  is separable (resp. fully faithful) and right adjoint of  $u_*$ .

Thus an adjoint pair of separable functors gives rise to a supplemented S- and  $S^\circ$ -category. In these notes (with a minor exception) we concentrate on S-categories.

### 3. EXAMPLES OF Q- AND S-CATEGORIES

The following generic example of a Q-category was constructed by Kontsevich and Rosenberg in [12].

**Example 3.1** (The Q-category of morphisms). Let  $\mathfrak{X}$  be any category, and let  $\mathfrak{X}^2$  be the category of morphisms in  $\mathfrak{X}$  defined as follows. The objects of  $\mathfrak{X}^2$  are morphisms  $f, g$  in  $\mathfrak{X}$ . Morphisms in  $\mathfrak{X}^2$  are commutative squares

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ \downarrow & & \downarrow \\ x' & \xrightarrow{g} & y' \end{array}$$

where the vertical arrows are in  $\mathfrak{X}$ . Now, set  $\tilde{\mathfrak{X}} = \mathfrak{X}^2$ . The inverse image functor  $u^*$  is

$$u^* : x \mapsto \left( x \xrightarrow{x} x \right), \quad \left( x \xrightarrow{f} y \right) \mapsto \left( \begin{array}{ccc} x & \xrightarrow{x} & x \\ f \downarrow & & \downarrow f \\ y & \xrightarrow{y} & y \end{array} \right).$$

The direct image functor  $u_*$  is defined by

$$u_* : \left( x \xrightarrow{f} y \right) \mapsto x, \quad \begin{array}{ccc} x & \xrightarrow{f} & y \\ \downarrow & & \downarrow \\ x' & \xrightarrow{g} & y' \end{array} \mapsto \left( \begin{array}{c} x \\ \downarrow \\ x' \end{array} \right).$$

Note that, for all objects  $x$  and morphisms  $f$  in  $\mathfrak{X}$ ,

$$u_* u^*(x) = u_*(x \xrightarrow{x} x) = x, \quad u_* u^*(f) = f.$$

Hence, for all objects  $x$  in  $\mathfrak{X}$ , there is an isomorphism (natural in  $x$ ),  $\eta_x : x \rightarrow u_* u^*(x)$ ,  $\eta_x = x$ .

Note further that for all objects  $x \xrightarrow{f} y$  in  $\mathfrak{X}^2$ ,  $u^* u_*(f) = x$ , and we can define a morphism  $\varepsilon_f : u^* u_*(f) \rightarrow f$  by

$$\varepsilon_f = \left( \begin{array}{ccc} x & \xrightarrow{x} & x \\ x \downarrow & & \downarrow f \\ x & \xrightarrow{f} & y \end{array} \right).$$

In this way,  $u_*$  is the right adjoint of  $u^*$  with counit  $\varepsilon$  and unit  $\eta$ . The unit is obviously a natural isomorphism, hence  $u^*$  is full and faithful and, thus, a Q-category  $\mathbb{X} = (\tilde{\mathfrak{X}} \xrightleftharpoons[u^*]{u_*} \mathfrak{X})$  is constructed.  $\mathbb{X}$  is supplemented, since  $u^*$  has a left adjoint

$$u_! : \left( x \xrightarrow{f} y \right) \mapsto y, \quad \left( \begin{array}{ccc} x & \xrightarrow{f} & y \\ \downarrow & & \downarrow \\ x' & \xrightarrow{g} & y' \end{array} \right) \mapsto \left( \begin{array}{c} y \\ \downarrow \\ y' \end{array} \right).$$

The unit of the adjunction  $u_! \dashv u^*$  is, for all  $f : x \rightarrow y$ ,

$$\bar{\eta}_f = \left( \begin{array}{ccc} x & \xrightarrow{f} & y \\ f \downarrow & & \downarrow y \\ y & \xrightarrow{y} & y \end{array} \right),$$

and thus the corresponding maps  $r$  come out as

$$r_f = f.$$

Consequently, an object  $x \in \mathfrak{X}$  is formally  $\mathbb{X}$ -smooth (when  $\mathbb{X}$  is supplemented by  $u_!$  and  $\bar{\eta}$ ) provided, for all  $y \xrightarrow{f} z \in \bar{\mathfrak{X}}$ , the mapping

$$\mathfrak{X}(x, y) \rightarrow \mathfrak{X}(x, z), \quad g \mapsto f \circ g,$$

is surjective. Similarly,  $x$  is formally  $\mathbb{X}$ -cosmooth if and only if the mappings

$$\mathfrak{X}(z, x) \rightarrow \mathfrak{X}(y, x), \quad g \mapsto g \circ f,$$

are surjective.

This generic example has a useful modification whereby one takes for  $\bar{\mathfrak{X}}$  any full subcategory of  $\mathfrak{X}^2$  which contains all the identity morphisms in  $\mathfrak{X}$ .

**Example 3.2** (The Wisbauer Q-category). Let  $R$  be a ring and  $M$  be a left  $R$ -module. Following [17, Section 15]  $\sigma[M]$  denotes a full subcategory of the category  ${}_R\mathfrak{M}$  of left  $R$ -modules, consisting of objects subgenerated by  $M$ . Since  $\sigma[M]$  is a full subcategory of  ${}_R\mathfrak{M}$ , the inclusion functor

$$u^* : \sigma[M] \rightarrow {}_R\mathfrak{M},$$

is full and faithful. It also has the right adjoint, the trace functor (see [17, 45.11] or [5, 41.1]),

$$u_* = \mathcal{T}^M : {}_R\mathfrak{M} \rightarrow \sigma[M], \quad \mathcal{T}^M(L) = \sum \{f(N) \mid N \in \sigma[M], f \in \text{Hom}_R(M, L)\}.$$

Hence there is a Q-category  $\mathbb{X} = (\bar{\mathfrak{X}} \xrightleftharpoons[u^*]{u_*} \mathfrak{X})$  with  $\mathfrak{X} = \sigma[M]$  and  $\bar{\mathfrak{X}} = {}_R\mathfrak{M}$ .

All the remaining examples come from the theory of corings.

**Example 3.3** (Comodules of a locally projective coring). This is a special case of Example 3.2. Let  $(\mathcal{C}, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$  be an  $A$ -coring which is locally projective as a left  $A$ -module. Let  $R = {}^*\mathcal{C} = \text{Hom}_{A-}(\mathcal{C}, A)$  be a left dual ring of  $\mathcal{C}$  with the unit  $\varepsilon_{\mathcal{C}}$  and product, for all  $r, s \in R$ ,

$$rs : \mathcal{C} \xrightarrow{\Delta_{\mathcal{C}}} \mathcal{C} \otimes_A \mathcal{C} \xrightarrow{\mathcal{C} \otimes_A s} \mathcal{C} \xrightarrow{r} A.$$

Take  $\mathfrak{X} = \mathfrak{M}^{\mathcal{C}}$ , the category of right  $\mathcal{C}$ -comodules, and  $\bar{\mathfrak{X}} = {}_R\mathfrak{M}$ . Define a functor

$$u^* : \mathfrak{M}^{\mathcal{C}} \rightarrow {}_R\mathfrak{M}, \quad M \mapsto M,$$

where right  $\mathcal{C}$ -comodule  $M$  is given a left  $R$ -module structure by  $rm = \sum m_{(0)}r(m_{(1)})$ . Since  $\mathcal{C}$  is a locally projective left  $A$ -module, the functor  $u^*$  has a right adjoint, the rational functor (see [5, 20.1]),

$$u_* = \text{Rat}^{\mathcal{C}} : {}_R\mathfrak{M} \rightarrow \mathfrak{M}^{\mathcal{C}}, \quad \text{Rat}^{\mathcal{C}}(M) = \{n \in M \mid n \text{ is rational}\},$$

where an element  $n \in M$  is said to be rational provided there exists  $\sum_i m_i \otimes_A c_i \in M \otimes_A \mathcal{C}$  such that, for all  $r \in R$ ,  $rm = \sum_i m_i r(c_i)$ . Here, the left  $R$ -module  $M$  is seen as a right  $A$ -module via the anti-algebra map  $A \rightarrow R$ ,  $a \mapsto \varepsilon_{\mathcal{C}}(-a)$ .

**Example 3.4** (Coseparable corings). Recall that an  $A$ -coring  $(\mathcal{C}, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$  is said to be *coseparable* [10] if there exists a  $(\mathcal{C}, \mathcal{C})$ -bicomodule retraction of the coproduct  $\Delta_{\mathcal{C}}$ . This is equivalent to the existence of a *cointegral* defined as an  $(A, A)$ -bimodule map  $\delta : \mathcal{C} \otimes_A \mathcal{C} \rightarrow A$  such that  $\delta \circ \Delta_{\mathcal{C}} = \varepsilon_{\mathcal{C}}$ , and

$$(\mathcal{C} \otimes_A \delta) \circ (\Delta_{\mathcal{C}} \otimes_A \mathcal{C}) = (\delta \otimes_A \mathcal{C}) \circ (\mathcal{C} \otimes_A \Delta_{\mathcal{C}}).$$

Furthermore, this is equivalent to the separability of the forgetful functor  $(-)_A : \mathfrak{M}^{\mathcal{C}} \rightarrow \mathfrak{M}_A$  [4, Theorem 3.5]). Since this forgetful functor is a left adjoint to  $- \otimes_A \mathcal{C} : \mathfrak{M}_A \rightarrow \mathfrak{M}^{\mathcal{C}}$ , a coseparable coring  $\mathcal{C}$  gives rise to an S-category  $\mathbb{X}$  with

$$\mathfrak{X} = \mathfrak{M}^{\mathcal{C}}, \quad \bar{\mathfrak{X}} = \mathfrak{M}_A, \quad u^* = (-)_A, \quad u_* = - \otimes_A \mathcal{C}.$$

This S-category is denoted by  $\mathbb{X}_{\delta}^{\mathcal{C}}$ . By [4, Theorem 3.5], the retraction  $\nu$  of the unit of the adjunction is given explicitly, for all  $M \in \mathfrak{M}^{\mathcal{C}}$ ,

$$\nu_M : M \otimes_A \mathcal{C} \rightarrow M, \quad m \otimes_A c \mapsto \sum m_{(0)} \delta(m_{(1)} \otimes_A c).$$

In general,  $\mathbb{X}_{\delta}^{\mathcal{C}}$  need not to be supplemented. However, if there exists

$$e \in \mathcal{C}^A := \{c \in \mathcal{C} \mid \forall a \in A, ac = ca\},$$

then  $\mathbb{X}_{\delta}^{\mathcal{C}}$  can be supplemented with

$$u_{\dagger} = - \otimes_A \mathcal{C}, \quad \bar{\eta}_M : M \rightarrow M \otimes_A \mathcal{C}, \quad m \mapsto m \otimes_A e.$$

This supplemented S-category is denoted by  $\mathbb{X}_{\delta, e}^{\mathcal{C}}$ .

Recall that an  $A$ -coring  $\mathcal{C}$  is said to be *cosplit* if there exists an  $A$ -central element  $e \in \mathcal{C}^A$  such that  $\varepsilon_{\mathcal{C}}(e) = 1$ . By [4, Theorem 3.3] this is equivalent to the separability of the functor  $- \otimes_A \mathcal{C}$ , and thus a cosplit coring gives rise to an  $S^{\circ}$ -category. Therefore, a coring which is both cosplit and coseparable induces a self-dual, supplemented S-category.

In addition to the defining adjunction of an  $A$ -coring,  $(-)_A \dashv - \otimes_A \mathcal{C}$ , for any right  $\mathcal{C}$ -comodule  $P$ , there is a pair of adjoint functors

$$- \otimes_B P : \mathfrak{M}_B \rightarrow \mathfrak{M}^{\mathcal{C}}, \quad \text{Hom}^{\mathcal{C}}(P, -) : \mathfrak{M}^{\mathcal{C}} \rightarrow \mathfrak{M}_B,$$

where  $B$  is any subring of the endomorphism ring  $S = \text{End}^{\mathcal{C}}(P)$  (cf. [5, 18.21]). Depending on the choice of  $\mathcal{C}$ ,  $P$  and  $B$  this adjunction provides a number of examples of Q-categories.

**Example 3.5** (Comatrix corings). Take a  $(B, A)$ -bimodule  $P$  that is finitely generated and projective as a right  $A$ -module. Let  $\mathbf{e} \in P \otimes_A P^*$  be the dual basis (where  $P^* = \text{Hom}_A(P, A)$ ), and let  $\mathcal{C} = P^* \otimes_B P$  be the comatrix coring associated to  $P$  [9]. The coproduct and counit in  $\mathcal{C}$  are given by

$$\Delta_{\mathcal{C}}(\xi \otimes_B p) = \xi \otimes_B \mathbf{e} \otimes_B p, \quad \varepsilon_{\mathcal{C}}(\xi \otimes_B p) = \xi(p),$$

for all  $p \in P$  and  $\xi \in P^*$ .  $P$  is a right  $\mathcal{C}$ -comodule with the coaction  $\varrho^P : p \mapsto \mathbf{e} \otimes_B p$ . Let

$$\mathfrak{X} = \mathfrak{M}_B, \quad \bar{\mathfrak{X}} = \mathfrak{M}^{\mathcal{C}}, \quad u^* = - \otimes_B P, \quad u_* = \text{Hom}^{\mathcal{C}}(P, -).$$

In view of [6, Proposition 2.3],  $\mathbb{X} = (\bar{\mathfrak{X}} \xrightleftharpoons[u^*]{u_*} \mathfrak{X})$  is a Q-category if and only if the map

$$B \rightarrow P \otimes_A P^*, \quad b \mapsto b\mathbf{e},$$

is pure as a morphism of left  $B$ -modules (equivalently,  $P$  is a totally faithful left  $B$ -module).

**Example 3.6** (Strongly  $(\mathcal{C}, A)$ -injective comodules). Let  $\mathcal{C}$  be an  $A$ -coring, let  $P$  be a right  $\mathcal{C}$ -comodule and  $S = \text{End}^{\mathcal{C}}(P)$ . Following [18, 2.9],  $P$  is said to be *strongly  $(\mathcal{C}, A)$ -injective* if the coaction  $\varrho^P : P \rightarrow P \otimes_A \mathcal{C}$  has a left  $S$ -module right  $\mathcal{C}$ -comodule retraction. For such a comodule, define

$$\mathfrak{X} = \mathfrak{M}_S, \quad \bar{\mathfrak{X}} = \mathfrak{M}^{\mathcal{C}}, \quad u^* = - \otimes_S P, \quad u_* = \text{Hom}^{\mathcal{C}}(P, -).$$

In view of [18, 3.2], if  $P$  is a finitely generated and projective as a right  $A$ -module, then  $\mathbb{X} = (\bar{\mathfrak{X}} \xrightleftharpoons[u^*]{u_*} \mathfrak{X})$  is a Q-category.

**Example 3.7** ( $(\mathcal{C}, A)$ -injective Galois comodules). Recall that a right  $\mathcal{C}$ -comodule is said to be  $(\mathcal{C}, A)$ -*injective*, provided there is a right  $\mathcal{C}$ -colinear retraction of the coaction. The full subcategory of  $\mathfrak{M}^{\mathcal{C}}$  consisting of all  $(\mathcal{C}, A)$ -injective comodules is denoted by  $\mathfrak{J}^{\mathcal{C}}$ .

Let  $P$  be a right comodule of an  $A$ -coring  $\mathcal{C}$ , and let  $S = \text{End}^{\mathcal{C}}(P)$  and  $T = \text{End}_A(P)$ . Following [18, 4.1],  $P$  is said to be a *Galois comodule* if, for all  $N \in \mathfrak{J}^{\mathcal{C}}$ , the evaluation map

$$\text{Hom}^{\mathcal{C}}(P, N) \otimes_S P \rightarrow N, \quad f \otimes_S p \rightarrow f(p),$$

is an isomorphism of right  $\mathcal{C}$ -comodules.

Let  $P$  be a Galois comodule, and assume that the inclusion  $S \rightarrow T$  has a right  $S$ -module retraction. By [18, 4.3] this is equivalent to say that  $P$  is a  $(\mathcal{C}, A)$ -injective comodule, and hence one can consider the following pair of categories and adjoint functors:

$$\bar{\mathfrak{X}} = \mathfrak{M}_S, \quad \mathfrak{X} = \mathfrak{J}^{\mathcal{C}}, \quad u_* = - \otimes_S P : \bar{\mathfrak{X}} \rightarrow \mathfrak{X}, \quad u^* = \text{Hom}^{\mathcal{C}}(P, -) : \mathfrak{X} \rightarrow \bar{\mathfrak{X}}.$$

Since the evaluation map is the counit of the adjunction  $u_* \dashv u^*$ , the Galois property of  $P$  means that the functor  $u^*$  is fully faithful. Thus  $\mathbb{X} = (\bar{\mathfrak{X}} \xrightleftharpoons[u^*]{u_*} \mathfrak{X})$  is a  $\text{Q}^{\circ}$ -category.

#### 4. EXAMPLES OF SMOOTH AND COSMOOTH OBJECTS

Let  $\mathcal{C}$  be an  $A$ -coring, set  $\mathfrak{X} = \mathfrak{M}^{\mathcal{C}}$ , and consider the full subcategory of  $\mathfrak{X}^2$  consisting of all monomorphisms in  $\mathfrak{M}^{\mathcal{C}}$  with an  $A$ -module retraction. With these data one constructs a Q-category as in Example 3.1. This Q-category is denoted by  $\mathbb{X}^{\mathcal{C}}$ .

**Theorem 4.1.** *A right  $\mathcal{C}$ -comodule  $M$  is  $(\mathcal{C}, A)$ -injective if and only if  $M$  is a formally  $\mathbb{X}^{\mathcal{C}}$ -cosmooth object.*

*Proof.* In view of the discussion at the end of Example 3.1, an object  $M \in \mathfrak{X} = \mathfrak{M}^{\mathcal{C}}$  is formally  $\mathbb{X}^{\mathcal{C}}$ -cosmooth if and only if, for all morphisms  $f : N \rightarrow N'$  in  $\mathfrak{M}^{\mathcal{C}}$  with right  $A$ -module retraction, the maps

$$\vartheta_f : \text{Hom}^{\mathcal{C}}(N', M) \rightarrow \text{Hom}^{\mathcal{C}}(N, M), \quad g \mapsto g \circ f,$$

are surjective. This means that, for all  $h \in \text{Hom}^{\mathcal{C}}(N, M)$ , there is  $g \in \text{Hom}^{\mathcal{C}}(N', M)$  completing the following diagram

$$\begin{array}{ccccc} & & M & & \\ & \nearrow h & & \nwarrow g & \\ 0 & \longrightarrow & N & \xrightleftharpoons{f} & N' \end{array}$$

where the arrow  $N' \rightarrow N$  is in  $\mathfrak{M}_A$ , and thus is equivalent to  $M$  being  $(\mathcal{C}, A)$ -injective, see [5, 18.18].  $\square$

The arguments used in the proof of Theorem 4.1, in particular, the identification of (co)smooth objects as object with a (co)splitting property, apply to all Q-categories of the type described in Example 3.1. This leads to reinterpretation of smooth algebras and coalgebras in abelian monoidal categories studied in [3].

**Example 4.2.** Let  $(V, \otimes)$  be an abelian monoidal category, i.e. a monoidal category which is abelian and such that the tensor functors  $- \otimes v, v \otimes -$  are additive and right exact, for all objects  $v$  of  $V$ . Let  $\mathfrak{X}$  be the category of algebras in  $V$ , and let  $\tilde{\mathfrak{X}}$  be a full subcategory of  $\mathfrak{X}^2$ , consisting of *Hochschild algebra extensions*, i.e. of all surjective algebra morphisms split as morphisms in  $V$  and with a square-zero kernel. Denote the resulting Q-category by  $\mathbb{H}\mathbb{A}\mathbb{E}$ . In view of [3, Theorem 3.8], an algebra in  $V$  is formally smooth in the sense of [3, Definition 3.9], i.e. it has the Hochschild dimension at most 1, if and only if it is a formally  $\mathbb{H}\mathbb{A}\mathbb{E}$ -smooth object.

In particular if  $(V, \otimes)$  is the category of vector spaces (with the usual tensor product), we obtain the characterisation of smooth algebras [16] (or semi-free algebras in the sense of [8]), described in [12, Proposition 4.3].

**Example 4.3.** Let  $(V, \otimes)$  be an abelian monoidal category. Let  $\mathfrak{X}$  be the category of coalgebras in  $V$ , and let  $\tilde{\mathfrak{X}}$  be a full subcategory of  $\mathfrak{X}^2$ , consisting of *Hochschild coalgebra extensions*, i.e. of all injective coalgebra morphisms  $\sigma : C \rightarrow E$  split as morphisms in  $V$  and with the property  $(p \otimes p) \circ \Delta_E = 0$ , where  $p : E \rightarrow \text{coker} \sigma$  is the cokernel of  $\sigma$ . Denote the resulting Q-category by  $\mathbb{H}\mathbb{C}\mathbb{E}$ . In view of [3, Theorem 4.16], a coalgebra in  $V$  is formally smooth in the sense of [3, Definition 4.17] if and only if it is a formally  $\mathbb{H}\mathbb{C}\mathbb{E}$ -cosmooth object.

The following example is taken from [2].

**Example 4.4.** Let  $A$  and  $B$  be rings, and let  $M$  be a  $(B, A)$ -bimodule. Denote by  $\mathcal{E}_M$  the class of all  $(B, B)$ -bilinear maps  $f$  such that  $\text{Hom}_B(M, f)$  splits as an  $(A, B)$ -bimodule map. A  $B$ -bimodule  $P$  is said to be  $\mathcal{E}_M$ -projective, provided every morphism  $N \rightarrow P$  in  $\mathcal{E}_M$  has a section. By the argument dual to that in the proof of Theorem 4.1 one can reinterpret  $\mathcal{E}_M$ -projectivity as formal smoothness as follows.

Take  $\mathfrak{X}$  to be the category of  $B$ -bimodules and  $\tilde{\mathfrak{X}} = \mathcal{E}_M$ , a full subcategory of  $\mathfrak{X}^2$ . Denote the resulting Q-category by  $\mathbb{E}$ . A  $B$ -bimodule  $P$  is formally  $\mathbb{E}$ -smooth if and only if, for all  $f : N \rightarrow N' \in \mathcal{E}_M$ , the function

$$\Theta(f) : \text{Hom}_{B,B}(P, N) \rightarrow \text{Hom}_{B,B}(P, N'), \quad g \mapsto f \circ g,$$

is surjective. In terminology of [11, Chapter X],  $\mathbb{E}$ -smoothness of  $P$  is equivalent to the  $\mathcal{E}_M$ -projectivity of  $P$ .

A  $(B, A)$ -bimodule  $M$  is said to be *formally smooth* provided the kernel of the evaluation map

$$\text{ev}_M : M \otimes_A \text{Hom}_B(M, B) \rightarrow B, \quad \text{ev}_M(m \otimes_A f) = f(m).$$

is an  $\mathcal{E}_M$ -projective  $B$ -bimodule. Thus  $M$  is formally smooth if and only if  $\ker \text{ev}_M$  is formally  $\mathbb{E}$ -smooth.

Next we characterise all smooth and cosmooth objects in the supplemented S-category  $\mathbb{X}_{\delta, e}^{\mathcal{C}}$  associated to a coseparable  $A$ -coring  $\mathcal{C}$  with an  $A$ -central element  $e$  as in Example 3.4.

**Proposition 4.5.** *Let  $\mathcal{C}$  be a coseparable  $A$ -coring with a cointegral  $\delta$  and an  $A$ -central element  $e$ , and let  $\mathbb{X}_{\delta,e}^{\mathcal{C}}$  be the associated supplemented  $S$ -category. A right  $\mathcal{C}$ -comodule  $M$  is formally  $\mathbb{X}_{\delta,e}^{\mathcal{C}}$ -smooth if and only if the map*

$$\kappa_M : M \rightarrow M, \quad m \mapsto \sum m_{(0)} \delta(e \otimes_A m_{(1)}),$$

*is a right  $A$ -linear section (i.e.  $\kappa_M$  has a left inverse in  $\text{End}_A(M)$ ).*

*Proof.* In this case, for all  $N \in \mathfrak{M}_A$ , the canonical morphisms  $r_N$  read

$$r_N : N \otimes_A \mathcal{C} \rightarrow N \otimes_A \mathcal{C}, \quad n \otimes_A c \mapsto \sum n \otimes_A e_{(1)} \delta(e_{(2)} \otimes_A c).$$

Using the (defining adjunction) isomorphisms  $\text{Hom}^{\mathcal{C}}(M, N \otimes_A \mathcal{C}) \simeq \text{Hom}_A(M, N)$ , the maps

$$\text{Hom}^{\mathcal{C}}(M, r_N) : \text{Hom}^{\mathcal{C}}(M, N \otimes_A \mathcal{C}) \rightarrow \text{Hom}^{\mathcal{C}}(M, N \otimes_A \mathcal{C}),$$

can be identified with

$$\vartheta_{M,N} : \text{Hom}_A(M, N) \rightarrow \text{Hom}_A(M, N), \quad f \mapsto (N \otimes_A \varepsilon_{\mathcal{C}}) \circ r_N \circ (f \otimes_A \mathcal{C}) \circ \varrho^M,$$

where  $\varrho^M : M \rightarrow M \otimes_A \mathcal{C}$  is the coaction. Hence  $\text{Hom}^{\mathcal{C}}(M, r_N)$  are surjective for all  $N$  if and only if  $\vartheta_{M,N}$  are surjective for all  $N$ . These can be computed further, for all  $m \in M$ ,  $f \in \text{Hom}_A(M, N)$ ,

$$\begin{aligned} \vartheta_{M,N}(f)(m) &= (N \otimes_A \varepsilon_{\mathcal{C}}) \circ r_N(f(m_{(0)}) \otimes_A m_{(1)}) \\ &= (N \otimes_A \varepsilon_{\mathcal{C}})(f(m_{(0)}) \otimes_A e_{(1)} \delta(e_{(2)} \otimes_A m_{(1)})) = \sum f(m_{(0)}) \delta(e \otimes_A m_{(1)}) \\ &= \sum f(m_{(0)} \delta(e \otimes_A m_{(1)})) = f(\kappa_M(m)), \end{aligned}$$

by the right  $A$ -linearity of  $f$ . Hence

$$\vartheta_{M,N}(f) = f \circ \kappa_M.$$

If  $\kappa_M$  has a retraction  $\lambda_M \in \text{End}_A(M)$ , then for all  $f \in \text{Hom}_A(M, N)$ ,

$$\vartheta_{M,N}(f \circ \lambda_M) = f \circ \lambda_M \circ \kappa_M = f,$$

i.e., the  $\vartheta_{M,N}$  are surjective. If, on the other hand, all the  $\vartheta_{M,N}$  are surjective, choose  $N = M$  and take any  $\lambda_M \in \vartheta_{M,M}^{-1}(M)$ . Then

$$M = \vartheta_{M,M}(\lambda_M) = \lambda_M \circ \kappa_M,$$

so  $\lambda_M$  is a retraction of  $\kappa_M$  as required.  $\square$

**Example 4.6** (Modules graded by  $G$ -sets). Let  $G$  be a group,  $X$  be a (right)  $G$ -set and let  $A = \bigoplus_{\sigma \in G} A_{\sigma}$  be a  $G$ -graded  $k$ -algebra. Following [13], a  $kX$ -graded right  $A$ -module  $M = \bigoplus_{x \in X} M_x$  is said to be *graded by  $G$ -set  $X$*  provided, for all  $x \in X$ ,  $\sigma \in G$ ,

$$M_x A_{\sigma} \subseteq M_{x\sigma}.$$

A morphism of such modules is an  $A$ -linear map which preserves the  $X$ -grading. The resulting category is denoted by  $\text{gr}(G, A, X)$ . It is shown in [7, Section 4.6] that  $\text{gr}(G, A, X)$  is isomorphic to the category of right comodules of the following coring  $\mathcal{C}$ . As a left  $A$ -module  $\mathcal{C} = A \otimes kX$ . The right  $A$ -multiplication is given by

$$(a \otimes x) a_{\sigma} = a a_{\sigma} \otimes x \sigma, \quad \forall a \in A, x \in X, a_{\sigma} \in A_{\sigma}.$$



The coproduct and counit are defined by

$$\Delta_{\mathcal{C}}(a \otimes x) = (a \otimes x) \otimes_A (1_A \otimes x), \quad \varepsilon_{\mathcal{C}}(a \otimes x) = a.$$

An object  $M = \bigoplus_{x \in X} M_x$  in  $\text{gr-}(G, A, X)$  is a right  $\mathcal{C}$ -comodule with the coaction  $\varrho^M : M \rightarrow M \otimes_A \mathcal{C}$ ,  $m_x \mapsto m_x \otimes_A 1_A \otimes x$ , where  $m_x \in M_x$ . Also in [7, Section 4.6] it is shown that  $\mathcal{C}$  is a coseparable coring with a cointegral (cf. [19, Proposition 2.5.3])

$$\delta : \mathcal{C} \otimes_A \mathcal{C} \simeq A \otimes kX \otimes kX \rightarrow A, \quad a \otimes x \otimes y \mapsto a \delta_{x,y}.$$

Thus  $\text{gr-}(G, A, X)$  gives rise to an S-category as in Example 3.4.

Let  $X^G := \{x \in X \mid \forall \sigma \in G, x\sigma = x\}$  be the set of one-point orbits of  $G$  in  $X$ . If  $X^G \neq \emptyset$ , the above S-category can be supplemented as in Example 3.4 by

$$e := 1_A \otimes z, \quad z \in X^G.$$

In this case, for any  $M \in \text{gr-}(G, A, X)$ , the map  $\kappa_M$  in Proposition 4.5 comes out as

$$\kappa_M(m_x) = m_x \delta_{x,z}, \quad \forall m_x \in M_x.$$

Thus a graded module  $M \in \text{gr-}(G, A, X)$  is formally  $\mathbb{X}_{\delta,e}^{A \otimes kX}$ -smooth if and only if it is concentrated in degree  $z$ , i.e.,  $M = M_z$ .

Given an  $A$ -coring  $\mathcal{C}$ , the set of right  $A$ -module maps  $\mathcal{C} \rightarrow A$ ,  $\mathcal{C}^*$ , is a ring with the unit  $\varepsilon_{\mathcal{C}}$  and the product, for all  $\xi, \xi' \in \mathcal{C}^*$ ,

$$\xi \xi' : \mathcal{C} \xrightarrow{\Delta_{\mathcal{C}}} \mathcal{C} \otimes_A \mathcal{C} \xrightarrow{\xi' \otimes_A \mathcal{C}} \mathcal{C} \xrightarrow{\xi} A.$$

**Proposition 4.7.** *Let  $\mathcal{C}$  be a coseparable  $A$ -coring with a cointegral  $\delta$  and an  $A$ -central element  $e$ , and let  $\mathbb{X}_{\delta,e}^{\mathcal{C}}$  be the associated supplemented S-category. Then the following statements are equivalent:*

- (1) *All right  $\mathcal{C}$ -comodules are formally  $\mathbb{X}_{\delta,e}^{\mathcal{C}}$ -cosmooth.*
- (2) *The right  $A$ -linear map*

$$\lambda : \mathcal{C} \rightarrow A, \quad c \mapsto \delta(e \otimes_A c),$$

*has a left inverse in the dual ring  $\mathcal{C}^*$ .*

- (3) *The regular right  $\mathcal{C}$ -comodule  $\mathcal{C}$  is formally  $\mathbb{X}_{\delta,e}^{\mathcal{C}}$ -cosmooth.*

*Proof.* Note that  $\mathcal{C}^*$  can be identified with  $\text{End}^{\mathcal{C}}(\mathcal{C})$  via the map  $\xi \mapsto (\xi \otimes_A \mathcal{C}) \circ \Delta_{\mathcal{C}}$  (with the inverse  $f \mapsto \varepsilon_{\mathcal{C}} \circ f$ ). Under this identification the product in  $\mathcal{C}^*$  coincides with the composition in  $\text{End}^{\mathcal{C}}(\mathcal{C})$ . Hence (2) is equivalent to saying that the map  $r_A = (\lambda \otimes_A \mathcal{C}) \circ \Delta_{\mathcal{C}}$  has a retraction in  $\mathfrak{M}^{\mathcal{C}}$ . Denote this retraction by  $s_A$ . Note further that, since  $\delta$  is a cointegral, the  $r_N$  defined in the proof of Proposition 4.5 can be written as  $r_N = N \otimes_A r_A$ . This implies that  $s_A$  is a section of  $r_A$  if and only if  $s_N = N \otimes_A s_A$  is a retraction of  $r_N = N \otimes_A r_A$ , for all right  $A$ -modules  $N$ . Finally observe that for all  $M \in \mathfrak{M}^{\mathcal{C}}$  and  $N \in \mathfrak{M}_A$ , the maps  $\varphi_{M,N} := \text{Hom}^{\mathcal{C}}(r_N, M)$  come out explicitly as

$$\varphi_{M,N} : \text{Hom}^{\mathcal{C}}(N \otimes_A \mathcal{C}, M) \ni f \mapsto f \circ r_N \in \text{Hom}^{\mathcal{C}}(N \otimes_A \mathcal{C}, M).$$

(2)  $\Rightarrow$  (1) The property  $s_N \circ r_N = N \otimes_A \mathcal{C}$ , implies that, for all right  $\mathcal{C}$ -comodules  $M$  and right  $A$ -modules  $N$ , the maps  $\varphi_{M,N}$  are surjective. Hence all right  $\mathcal{C}$ -comodules are formally  $\mathbb{X}_{\delta,e}^{\mathcal{C}}$ -cosmooth.

The implication (1)  $\Rightarrow$  (3) is obvious.

(3)  $\Rightarrow$  (2) If  $\mathcal{C}$  is formally  $\mathbb{X}_{\delta,e}^{\mathcal{C}}$ -cosmooth, then  $\varphi_{\mathcal{C},A} : \text{End}^{\mathcal{C}}(\mathcal{C}) \rightarrow \text{End}^{\mathcal{C}}(\mathcal{C})$  is surjective. Hence there exists  $s_A \in \text{End}^{\mathcal{C}}(\mathcal{C})$  such that

$$\mathcal{C} = \varphi_{\mathcal{C},A}(s_A) = s_A \circ r_A.$$

This completes the proof.  $\square$

A coseparable  $A$ -coring  $\mathcal{C}$  with a cointegral  $\delta$  is said to be *Frobenius-coseparable* if there exists  $e \in \mathcal{C}^A$  such that, for all  $c \in \mathcal{C}$ ,  $\delta(c \otimes_A e) = \delta(e \otimes_A c) = \varepsilon_{\mathcal{C}}(c)$ . The element  $e$  is called a *Frobenius element*. In particular a Frobenius-coseparable coring is a Frobenius coring, see [5, 27.5].

**Corollary 4.8.** *Let  $\mathcal{C}$  be a Frobenius-coseparable  $A$ -coring with cointegral  $\delta$  and Frobenius element  $e$ . Then any right  $\mathcal{C}$ -comodule is formally  $\mathbb{X}_{\delta,e}^{\mathcal{C}}$ -cosmooth and  $\mathbb{X}_{\delta,e}^{\mathcal{C}}$ -smooth.*

*Proof.* The maps  $\kappa_M$  in Proposition 4.5 are all identity morphisms, hence they are sections and thus every right  $\mathcal{C}$ -comodule is formally  $\mathbb{X}_{\delta,e}^{\mathcal{C}}$ -smooth. The map  $\lambda$  in Proposition 4.7 coincides with the counit  $\varepsilon_{\mathcal{C}}$ . Since  $\varepsilon_{\mathcal{C}}$  is a unit in  $\mathcal{C}^*$ , it has a left inverse, and thus every right  $\mathcal{C}$ -comodule is formally  $\mathbb{X}_{\delta,e}^{\mathcal{C}}$ -cosmooth.  $\square$

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